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Localized direct boundary-domain integro-differential formulations for scalar nonlinear boundary-value problems with variable coefficients

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Abstract. Mixed boundary-value Problems (BVPs) for a second-order quasi-linear elliptic partial differential equation with variable coefficients dependent on the unknown solution and its gradient are considered. Localized parametrices of auxiliary linear partial differential equations along with different combinations of the Green identities for the original and auxiliary equations are used to reduce the BVPs to direct or two-operator direct quasi-linear localized boundary-domain integro-differential equations (LBDIDEs). Different parametrix localizations are discussed, and the corresponding nonlinear LBDIDEs are presented. Mesh-based and mesh-less algorithms for the LBDIDE discretization are described that reduce the LBDIDEs to sparse systems of quasi-linear algebraic equations.

Key words: compressible flow, heat transfer, integro-differential equations, mesh-based and mesh-less algorithms, partial differential equations

1. Introduction

It is well-known that a boundary-value problem (BVP) for a non-linear partial differential equation (PDE) can be reduced to a non linear boundary-domain integral equation (BDIE), see *e.g.* [1, Chapters 7, 8; Section 12.6], [2, Chapter 6], [3, Chapter 13, 15], [4, Section 8.9] [5, Chapter 6], using the fundamental solution of an auxiliary linear PDE with coefficients evaluated either for zero or for the current value of the unknown variable in the source point. However, the fundamental solution is generally not available in an explicit and/or cheaply computable form if the coefficients of the auxiliary PDE depend on the space variables. Moreover, the fundamental solution of the auxiliary PDE is usually highly non-local, which leads, after discretization, to a system of nonlinear algebraic equations with a fully populated matrix.

To prevent such difficulties, *localized* parametrices were constructed in [6], reducing a *linear* elliptic BVP with variable coefficients to a direct linear *localized* boundary-domain integral equation (LBDIE). Some numerical implementations of the linear LBDIE were presented in [7]. Following [8], this method is generalized in Section 2 to reduce a mixed BVP for a second-order *quasi-linear* elliptic PDE with variable coefficients, dependent also on the unknown solution, to direct *quasi-linear* single-operator localized boundary-domain integro-differential equations (LBDIDEs). However, if the coefficients of the BVP depend not only on the unknown solution but also on its gradient, the single-operator approach leads to LBDIDEs involving second-order derivatives. To obtain a direct LBDIDE with first derivatives at most, a two-operator Green identity for the original and an auxiliary PDE is derived in Section 3, following [9]. In principle, one could then reduce the single-operator, as well as the two-operator direct LBDIDEs, to nonlinear boundary-domain *integral* equations (involving Cauchy-singular integrals over the domain and hyper-singular integrals over the boundary), using the integral representations for the solution gradients considered as separate unknown variables similar to [1, Chapter 7], [2, Chapter 6], [3, Chapter 13], [5, Chapter 6]. We will not follow this route and describe instead in Section 4 the straightforward discretization of the LBDIDEs, employing either a mesh-based or a mesh-less collocation approach and the corresponding solution approximation in terms of the nodal values. Both discretizations reduce the LBDIDEs to sparse systems of quasi-linear algebraic equations.

2. Direct integro-differential formulations

To illustrate the general approach of reducing a mixed BVP for a second-order quasi-linear elliptic PDE with variable coefficients dependent on the unknown solution to direct LBDIDEs, we consider in this section the mixed BVP of stationary nonlinear heat transfer in an isotropic inhomogeneous medium.

2.1. Nonlinear BVP of stationary heat transfer in an inhomogeneous body and Green's identity

Let us consider a body occupying an open domain, $\Omega\subset\mathbb{R}^n$, where $n=2$ or $n=3$, with a prescribed temperature $\bar{u}(x)$ on a closed part $\partial_D\Omega$ of the boundary $\partial\Omega$ and prescribed heat flux $\bar{t}(x)$ on the remaining open part $\partial_N\Omega$,

$$[L(u)u](x):=\frac{\partial}{\partial x_i}\left[a(u(x),x)\frac{\partial u(x)}{\partial x_i}\right]=f(x),\quad x\in\Omega, \tag{1}$$

$$u(x)=\bar{u}(x),\quad x\in\partial_D\Omega, \tag{2}$$

$$[T(u)u](x):=a(u(x),x)\frac{\partial u(x)}{\partial n(x)}=\bar{t}(x),\quad x\in\partial_N\Omega. \tag{3}$$

Here $u(x)$ is the unknown temperature, $[L(\lambda)u](x):=\frac{\partial}{\partial x_i}\left[a(\lambda(x),x)\frac{\partial u(x)}{\partial x_i}\right]$ is a linear differential operator, $[T(\lambda)u](x):=a(\lambda(x),x)\partial u(x)/\partial n(x)$ is a linear surface-flux operator and $a(\lambda(x),x)>C>0$ is a variable thermo-conductivity coefficient dependent on a function $\lambda(x)$, $f(x)$ is a known distributed heat source, $n(x)$ is the outward unit normal vector to the boundary $\partial\Omega$, $\bar{u}(x)$ and $\bar{t}(x)$ are known functions. Summation over repeated indices is assumed from 1 to 2 in the 2D case, and from 1 to 3 in the 3D case, unless stated otherwise. BVP (1–3) becomes a pure Neumann problem if $\partial_D\Omega=\emptyset$, and a pure Dirichlet problem if $\partial_N\Omega=\emptyset$. Note that the well-known Kirchhoff transform (see *e.g.* [4, Section 4.6]) cannot be used to linearize this problem, since $a(u(x),x)$ depends not only on the unknown variable u but also on the coordinate x.

The second Green identity for the differential operator $L(u)$ takes the form

$$\int_\Omega\{u(x)[L(u)v](x)-v(x)[L(u)u](x)\}\,\mathrm{d}\Omega(x)=\int_{\partial\Omega}\{u(x)[T(u)v](x)-v(x)[T(u)u](x)\}\,\mathrm{d}\Gamma(x), \tag{4}$$

where $u(x)$ and $v(x)$ are arbitrary functions ensuring that the integrals make sense (either classical or distributional).

If $L(u)$ is a linear operator, $L(u)=L$, and $F(x,y)$ is its fundamental solution, *i.e.*,

$$[LF(\cdot,y)](x)=\delta(x-y),$$

where $\delta(x-y)$ is the Dirac delta-function, then one could take $v(x)=F(x,y)$, identify $u(x)$ with a solution of Equation (1), and thus arrive at the third Green identity,

$$c(y)u(y)-\int_{\partial\Omega}\{u(x)[TF(\cdot,y)](x)-F(x,y)[Tu](x)\}\mathrm{d}\Gamma(x)=\int_{\Omega}F(x,y)f(x)\mathrm{d}\Omega(x),\quad y\in\mathbb{R}^n, \tag{5}$$

$$c(y)=c(y;\Omega)=\begin{cases}1 & \text{if } y\in\Omega,\\ 0 & \text{if } y\notin\bar{\Omega},\\ \alpha(y;\Omega)/(2\pi) & \text{if } y\in\partial\Omega \quad\text{and}\quad \Omega\subset\mathbb{R}^3,\\ \alpha(y;\Omega)/(4\pi) & \text{if } y\in\partial\Omega \quad\text{and}\quad \Omega\subset\mathbb{R}^3,\end{cases} \tag{6}$$

where $\alpha(y;\Omega)$ is the interior solid angle at a corner point y of the boundary $\partial\Omega$; in particular, $c(y)=1/2$ if y is a smooth point of the boundary. Substituting the boundary conditions in the Green identity (5) and applying it for $y\in\partial\Omega$, we arrive at a direct boundary-integral equation; see *e.g.* [1, Section 2.4].

2.2. Parametrix and quasi-linear direct integro-differential equations

For the partial differential operator $L(\lambda)$ with a variable coefficient $a(\lambda(x),x)$, a fundamental solution is generally not available in explicit form. Instead, however, a *parametrix* $P(\lambda;x,y)$ can be defined as a function of x, y and λ, such that

$$[L(\lambda)P(\lambda;\cdot,y)](x)=\delta(x-y)+R(\lambda;x,y),$$

where the remainder term $R(\lambda;x,y)$ is at most weakly singular (*i.e.*, integrable with respect to $x\in\Omega$), which is always available.

For a given operator $L(\lambda)$, the parametrix is evidently not unique. A particular parametrix $P(\lambda;x,y)$ is given by a fundamental solution $F^{(y)}(\lambda;x,y)=F(\lambda(y),x,y)$ of the corresponding operator with "frozen" coefficient,

$$[L^{(y)}(\lambda)v](x):=\frac{\partial}{\partial x_i}\left[a(\lambda(y),y)\frac{\partial v(x)}{\partial x_i}\right].$$

Evidently, $F(\lambda(y),x,y)=F_{\Delta}(x,y)/a(\lambda(y),y)$, where $F_{\Delta}(x,y)$ is a fundamental solution of the Laplace operator. Thus, denoting $|x-y|=\sqrt{(x_i-y_i)(x_i-y_i)}$, we can take,

$$\text{2D:}\qquad P(\lambda;x,y)=P(\lambda(y),x,y)=\frac{\log|x-y|}{2\pi a(\lambda(y),y)}, \tag{7}$$

$$R(\lambda;x,y)=R(\lambda(x),\lambda(y),\nabla\lambda(x),x,y)$$
$$=\frac{x_i-y_i}{2\pi a(\lambda(y),y)|x-y|^2}\left[\frac{\partial a(\lambda,x)}{\partial\lambda}\frac{\partial\lambda(x)}{\partial x_i}+\frac{\partial a(\lambda,x)}{\partial x_i}\right]_{\lambda=\lambda(x)}, \tag{8}$$

$$\text{3D:}\qquad P(\lambda;x,y)=P(\lambda(y),x,y)=\frac{-1}{4\pi a(\lambda(y),y)|x-y|}, \tag{9}$$

$$R(\lambda;x,y)=R(\lambda(x),\lambda(y),\nabla\lambda(x),x,y)$$
$$=\frac{x_i-y_i}{4\pi a(\lambda(y),y)|x-y|^3}\left[\frac{\partial a(\lambda,x)}{\partial\lambda}\frac{\partial\lambda(x)}{\partial x_i}+\frac{\partial a(\lambda,x)}{\partial x_i}\right]_{\lambda=\lambda(x)}. \tag{10}$$

Identifying $u(x)$ with a solution of PDE (1), assuming that $\lambda(x)=u(x)$, using $P(u;x,y)$ as $v(x)$ in Green's second identity (4), and employing the usual limiting procedure at y (see *e.g.*

[10, Section I.9]) similar to that for the fundamental solution, we arrive at the parametrix-based nonlinear counterpart of Green's third identity (5),

$$c(y)u(y) - \int_{\partial\Omega} \{u(x)[T(u)P(u;\cdot,y)](x) - P(u;x,y)[T(u)u](x)\}\,\mathrm{d}\Gamma(x) + \int_{\Omega} R(u;x,y)u(x)\mathrm{d}\Omega(x) = \int_{\Omega} P(u;x,y)f(x)\mathrm{d}\Omega(x), \qquad y \in \mathbb{R}^n, \tag{11}$$

where $c(y)$ is given by (5). As one can see from (8) and (10), the remainder $R(u;x,y)$ in (11) does depend not only on the values of solution u but also on its gradient ∇u.

Identity (11) can be used for formulating different boundary-domain integro-differential equations with respect to u and its derivatives. We consider below some of the formulations.

2.2.1. *United formulation*

We can substitute boundary conditions (2) and (3) in the integrals in (11) and use (11) at $y \in \bar{\Omega} = \Omega \cup \partial\Omega$, to reduce BVP (1–3) to the quasi-linear direct *boundary-domain integro-differential equation*, BDIDE, for $u(x)$ at $x \in \bar{\Omega}$,

$$c(y)u(y) - \int_{\partial_N\Omega} u(x)[T(u)P(u;\cdot,y)](x)\mathrm{d}\Gamma(x) + \int_{\partial_D\Omega} [T(u)u](x)P(u;x,y)(x)\mathrm{d}\Gamma(x) + \int_{\Omega} R(u;x,y)u(x)\mathrm{d}\Omega(x) = \mathcal{F}(u;y), \tag{12}$$

$$\mathcal{F}(u;y) := \int_{\partial_D\Omega} \bar{u}(x)[T(u)P(u;\cdot,y)](x)\mathrm{d}\Gamma(x) - \int_{\partial_N\Omega} P(u;x,y)\bar{t}(x)\mathrm{d}\Gamma(x) + \int_{\Omega} P(u;x,y)f(x)\mathrm{d}\Omega(x), \quad y \in \bar{\Omega}.$$

The second-kind form of BDIDE (12) looks attractive for constructing iterative algorithms of its numerical solution.

2.2.2. *Partly segregated formulation*

In a slightly different approach, we apply (12) at $y \in \bar{\Omega}$, substitute $\bar{u}(y)$ for $u(y)$ also in the out-of-integral term when $y \in \partial_D\Omega$ and introduce a new variable $t(x)$ for the unknown boundary flux $[T(u)u](x)$ on $\partial_D\Omega$. This reduces BVP (1–3) to another quasi-linear direct boundary-domain integro-differential equation, BDIDE, for $u(x)$ at $x \in \Omega \cup \partial_N\Omega$ and $t(x)$ at $x \in \partial_D\Omega$,

$$c^0(y)u(y) - \int_{\partial_N\Omega} u(x)[T(u)P(u;\cdot,y)](x)\mathrm{d}\Gamma(x) + \int_{\partial_D\Omega} t(x)P(u;x,y)\mathrm{d}\Gamma(x) + \int_{\Omega} R(u;x,y)u(x)\mathrm{d}\Omega(x) = \mathcal{F}^0(u;y), \tag{13}$$

$$\mathcal{F}^0(u;y) := [c^0(y) - c(y)]\bar{u}(y) + \mathcal{F}(u;y), \quad y \in \Omega \cup \partial\Omega, \tag{14}$$

$$c^0(y) = 0 \quad \text{if } y \in \partial_D\Omega, \quad c^0(y) = c(y) \quad \text{if } y \in \Omega \cup \partial_N\Omega. \tag{15}$$

We will consider the unknown boundary variable t on $\partial_D\Omega$ as formally segregated from the internal field u, that is, we will not use its relation to the boundary flux $[T(u)u](x)$, while solving (13).

Even for boundary points y, the domain integrals in (12–14) include the unknown values of u over the whole domain Ω. This prevents us from reducing the BDIDEs to a *boundary* integral equations for $u(x)$ on $\partial_N\Omega$ and $t(x)$ on $\partial_D\Omega$, as in the case when the parametrix is a fundamental solution.

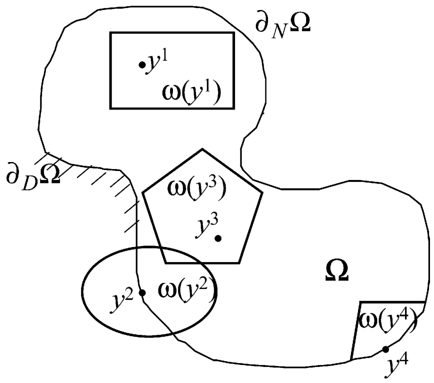

Figure 1. A body Ω with localization domains $\omega(y^i)$.

Evidently, the united and partly segregated formulations coincide for the pure Neumann problem, *i.e.*, when $\partial_D\Omega = \emptyset$. Equations (12) and (13) are integro-differential, since they both include dependence on the gradient ∇u in R, and BDIDE (12) includes the differential flux operator $T(u)u$ on $\partial_D\Omega$ as well. Note that not only the left-hand sides of BDIDEs (12) and (13) but also their right-hand sides $\mathcal{F}$ and $\mathcal{F}^0$ do depend on the unknown solution, u. Because of this and the dependence of the functions P, R and operator T on u, the BDIDEs are nonlinear. We call them quasi-linear for the form resembling their linear counterparts. If the original BVP (1–3) is linear, *i.e.*, the coefficient a does not depend on u, then T, P, R, $\mathcal{F}$ and $\mathcal{F}^0$ depend neither on u nor on ∇u, and BDIDEs (12) and (13) degenerate into the linear BDIDE and BDIE, respectively, with the known right-hand sides $\mathcal{F}$ and $\mathcal{F}^0$; *c.f.* [6] where the linear analog of (13) is presented.

2.3. Localized parametrices and direct BDIDEs

Although a parametrix is not unique, all parametrices $P(\lambda; x, y)$ of a differential operator $L(\lambda)$ exhibit the same singularity at $x = y$ but can differ at other points. Thus, we can perturb an available (not localized) parametrix $P(\lambda; x, y)$ to localize it. Particularly, we can consider $P_\omega(\lambda; x, y) = \chi(x, y)P(\lambda; x, y)$, where $\chi(x, y)$ is a cut-off function, such that $\chi(y, y) = 1$ and $\chi(x, y) = 0$ at x do not belong to a closed localization domain $\bar{\omega}(y)$, where y belongs to the corresponding open domain $\omega(y)$ or to its boundary $\partial\omega(y)$, as shown in Figure 1.

Then $P_\omega(\lambda; x, y)$ possesses the same singularity as $P(\lambda; x, y)$ at $x = y$ but is localized (non-zero) only in $\omega(y)$. Further we have,

$$\begin{aligned}
&[L(\lambda)P_\omega(\lambda; \cdot, y)](x) = [L_x(\lambda)\{\chi(\cdot, y)P(\lambda; \cdot, y)\}](x)\\
&\quad = [L(\lambda)P(\lambda; \cdot, y)](x) - [L(\lambda)\{(1 - \chi(\cdot, y))P(\lambda; \cdot, y)\}](x) = \delta(x - y) + R_\omega(\lambda; x, y),\\
&R_\omega(\lambda; x, y) = R(\lambda; x, y) - [L(\lambda)\{(1 - \chi(\cdot, y))P(\lambda; \cdot, y)\}](x).
\end{aligned}$$

Consequently, R_ω will have the necessary properties of the remainder, that is, $P_\omega(\lambda; x, y)$ is also a parametrix, at least if χ is sufficiently smooth.

2.3.1. *Discontinuous localization*

Let the localization domain $\omega(y)$ be an open domain, $y \in \bar{\omega}(y)$, and $\chi(x, y)$ be piece-wise continuous in $\mathbb{R}^n$,

$$\chi(x, y) = \begin{cases} \chi^1(x, y) & \text{if } x \in \bar{\omega}(y),\\ 0 & \text{if } x \notin \bar{\omega}(y), \end{cases} \tag{16}$$

where $\chi^1(x,y)$ is a smooth function in $x\in\bar{\omega}(y)$ such that $\chi^1(y,y)=1$. Then

$$P_\omega(\lambda;x,y)=\chi(x,y)P(\lambda;x,y)=\begin{cases}\chi^1(x,y)P(\lambda;x,y) & \text{if } x\in\bar{\omega}(y),\\ 0 & \text{if } x\notin\bar{\omega}(y)\end{cases} \tag{17}$$

is a discontinuous localized parametrix.

The simplest example of the cut-off function is the piecewise constant,

$$\chi(x,y)=\begin{cases}1 & \text{if } x\in\bar{\omega}(y)\\ 0 & \text{if } x\notin\bar{\omega}(y)\end{cases},\quad P_\omega(\lambda;x,y)=\begin{cases}P(\lambda;x,y) & \text{if } x\in\bar{\omega}(y),\\ 0 & \text{if } x\notin\bar{\omega}(y).\end{cases} \tag{18}$$

Assume that y lies either inside the domain $\omega(y)$ or on the intersection of the boundaries of the localization and global domains, $\partial\omega(y)\cap\partial\Omega$, such that $\alpha(y;\Omega)=\alpha(y;\omega(y))$. Substituting $P_\omega(u;x,y)$ from (17) for $v(x)$ in the second Green identity for the intersection of $\bar{\Omega}$ with $\bar{\omega}(y)$ and taking $u(x)$ as a solution to (1), we arrive at the third Green identity with integrals localized on $\bar{\omega}(y)\cap\bar{\Omega}$,

$$\begin{aligned}c(y)u(y)&-\int_{\bar{\omega}(y)\cap\partial\Omega}u(x)[T(u)P_\omega(u;\cdot,y)](x)\mathrm{d}\Gamma(x)+\int_{\bar{\omega}(y)\cap\partial\Omega}P_\omega(u;x,y)[T(u)u](x)\mathrm{d}\Gamma(x)\\&-\int_{\Omega\cap\partial\omega(y)}u(x)[T(u)P_\omega(u;\cdot,y)](x)\mathrm{d}\Gamma(x)+\int_{\Omega\cap\partial\omega(y)}P_\omega(u;x,y)[T(u)u](x)\mathrm{d}\Gamma(x)\\&+\int_{\omega(y)\cap\Omega}R_\omega(u;x,y)u(x)\mathrm{d}\Omega(x)=\int_{\omega(y)\cap\Omega}P_\omega(u;x,y)f(x)\mathrm{d}\Omega(x),\quad y\in\mathbb{R}^n,\end{aligned} \tag{19}$$

where $c(y)=c(y;\Omega)$ is given by the same formula (5).

United formulation. We can now substitute (2) and (3) in the first and the second integral terms of the left-hand side of equality (19) and use it at $y\in\bar{\Omega}$, thus arriving at the following quasi-linear direct LBDIDE,

$$\begin{aligned}c(y)u(y)&-\int_{\bar{\omega}(y)\cap\partial_N\Omega}u(x)[T(u)P_\omega(u;\cdot,y)](x)\mathrm{d}\Gamma(x)+\int_{\bar{\omega}(y)\cap\partial_D\Omega}P_\omega(u;x,y)[T(u)u](x)\mathrm{d}\Gamma(x)\\&-\int_{\Omega\cap\partial\omega(y)}u(x)[T(u)P_\omega(u;\cdot,y)](x)\mathrm{d}\Gamma(x)+\int_{\Omega\cap\partial\omega(y)}P_\omega(u;x,y)[T(u)u](x)\mathrm{d}\Gamma(x)\\&+\int_{\omega(y)\cap\Omega}R_\omega(u;x,y)u(x)\mathrm{d}\Omega(x)=\mathcal{F}_\omega(u;y),\quad y\in\bar{\Omega},\end{aligned} \tag{20}$$

$$\begin{aligned}\mathcal{F}_\omega(u;y):=&\int_{\bar{\omega}(y)\cap\partial_D\Omega}\bar{u}(x)[T(u)P_\omega(u;\cdot,y)](x)\mathrm{d}\Gamma(x)-\int_{\bar{\omega}(y)\cap\partial_N\Omega}P_\omega(u;x,y)\bar{t}(x)\mathrm{d}\Gamma(x)\\&+\int_{\omega(y)\cap\Omega}P_\omega(u;x,y)f(x)\mathrm{d}\Omega(x).\end{aligned} \tag{21}$$

Partly segregated formulation. Alternatively, substitution of $\bar{u}(y)$ also for the out-of-integral term $u(y)$ at $y\in\partial\Omega_D$ and introduction of a new variable $t(x)=[T(u)u](x)$ for the unknown flux at $x\in\partial\Omega_D$ in (20) reduces BVP (1–3) to the following partly segregated quasi-linear direct

LBDIDE for $u(x)$ at $x \in \Omega \cup \partial_N \Omega$ and $t(x)$ at $x \in \partial_D \Omega$,

$$c^0(y)u(y) - \int_{\bar{\omega}(y)\cap\partial_N\Omega} u(x)[T(u)P_\omega(u;\cdot,y)](x)\mathrm{d}\Gamma(x) + \int_{\bar{\omega}(y)\cap\partial_D\Omega} P_\omega(u;x,y)t(x)\mathrm{d}\Gamma(x)$$
$$- \int_{\Omega\cap\partial\omega(y)} u(x)[T(u)P_\omega(u;\cdot,y)](x)\mathrm{d}\Gamma(x) + \int_{\Omega\cap\partial\omega(y)} P_\omega(u;x,y)[T(u)u](x)\mathrm{d}\Gamma(x)$$
$$+ \int_{\omega(y)\cap\Omega} R_\omega(u;x,y)u(x)\mathrm{d}\Omega(x) = \mathcal{F}^0_\omega(u;y), \quad y \in \bar{\Omega}, \tag{22}$$

$$\mathcal{F}^0_\omega(u;y) := [c^0(y) - c(y)]\bar{u}(y) + \mathcal{F}_\omega(u;y), \tag{23}$$

where $c^0(y)$ is given by (15) and $\mathcal{F}_\omega$ by (21).

Not only the left-hand sides but also the right-hand sides, $\mathcal{F}_\omega(u;y)$ and $\mathcal{F}^0_\omega(u;y)$, of LBDIDEs (20) and (22) depend on the unknown function $u(x)$, $x \in \bar{\omega}(y) \cap \bar{\Omega}$.

As discussed in [6] for the linear case, BDIDEs (20) and (22) can also be interpreted as a domain-decomposition method, if a finite number of the localization domains ω covers the whole body Ω and the localization domains do not change during the discretization refinement but the point y is allowed to vary inside the corresponding domain $\omega \ni y$.

Although more general cut-off functions (*e.g.*, given by functions χ^1 in (16), which are piece-wise smooth in $\bar{\omega}(y)$, *c.f.* [6]) might also be considered, we will concentrate in this paper mainly on cut-off functions that are piece-wise continuous in $\mathbb{R}^n$ but smooth in $\bar{\omega}(y)$. The general integral equality (19) and LBDIDEs (20), (22) will be simplified for special choices of $\chi(x,y)$.

2.3.2. *Continuous localizations*

To get rid of the integrals involving $T(u)u$ on $\partial\omega(y)$, *i.e.*, the fourth integrals on the left-hand sides of (19), (20) and (22), one can construct a localized parametrix $P_\omega(u;x,y)$ that vanishes on the boundary $\partial\omega(y)$.

The Green function for a corresponding BVP with "frozen" constant coefficients in the differential operator L on $\omega(y)$ was employed in [11,12] as a parametrix $P_\omega(x,y)$ vanishing on $\partial\omega(y)$. However, the Green function is available in analytical form only for sufficiently simple shapes of the localization domain $\omega(y)$, *e.g.*, for a ball.

It seems simpler and more universal to use the cut-off approach and construct a proper localized parametrix as $P_\omega(\lambda;x,y) = \chi(x,y)P(\lambda;x,y)$. Here P is an available parametrix (*e.g.*, a fundamental solution for a corresponding differential operator with "frozen" coefficients) and a cut-off function $\chi(x,y)$ is smooth in $x \in \bar{\omega}(y)$ and equal to zero both on the boundary and outside $\omega(y)$. Then, evidently, $\chi(x,y)$ is continuous in $x \in \mathbb{R}^n$.

Some examples of such cut-off functions $\chi(x,y)$ localized on a ball or on a cube with y in its centre were presented in [6]. Here we give an example of $\chi(x,y)$ localized on a polyhedron ω^p with p sides s_j, $j = 1, 2, ..., p$. Let y be an internal point of a non-concave polyhedron, as shown in Figure 2. Then $\chi(x,y)$ can be taken as the product

$$\chi(x,y) = \prod_{j=1}^{p} \frac{\rho_j(x)}{\rho_j(y)}, \quad x \in \bar{\omega}^p(y),$$

where $\rho_i(x)$ is the distance of point x from the side s_j of the polyhedron. Here $\chi(x,y) = 0$ for $x \notin \bar{\omega}^p(y)$.

For $y \in \partial\Omega$, one can take a localization domain $\omega(y)$ only partly intersecting $\bar{\Omega}$, like $\omega(y^2)$ in Figure 1, and work further with the LBDIDEs in the intersection.

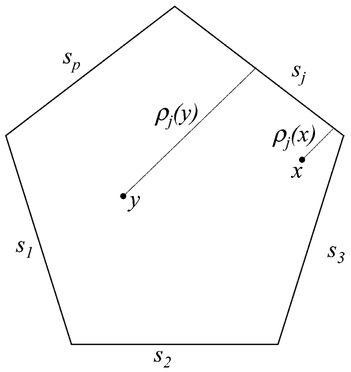

Figure 2. Example of a polyhedral localization domain $\omega^p(y)$ with an internal point y.

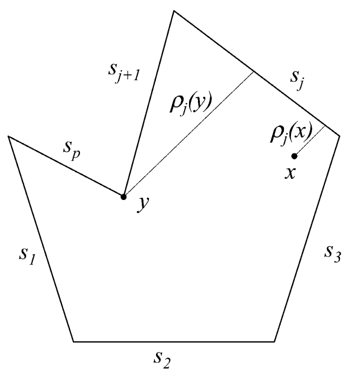

Figure 3. Example of a polyhedral localization domain $\omega^p(y)$ with a boundary point y.

Another option is to use localization domains $\omega(y)$ belonging to Ω, like $\omega(y^4)$ in Figure 1, for boundary points $y\in\partial\omega(y)\cap\partial\Omega$. To ensure $\chi(y,y)=1$, one should demand $\chi(x,y)=0$ not for all $x\in\partial\omega(y)$, but only for x on a part of $\partial\omega(y)$ not including a neighbourhood of $y\in\partial\omega(y)$. An example of such a cut-off function for a polyhedron $\omega^p(y)$ is

$$\chi(x,y)=\prod_{\bar{s}_j\not\ni y}\frac{\rho_j(x)}{\rho_j(y)},\quad x\in\bar{\omega}^p(y),$$

and $\chi(x,y)=0$ for $x\notin\bar{\omega}^p(y)$. In this case, one can relax the nonconcavity condition on the polyhedron for the sides that y belongs to (see Figure 3, where an extreme case is shown, when y belongs to a vertex, that is, to several sides of the polyhedron).

To consider that way of localization for the case $y\in\partial\omega(y)\cap\partial\Omega$ as continuous, one may continue the cut-off function $\chi(x,y)$ through $\partial\omega(y)\cap\partial\Omega$ outside Ω into a larger localization domain $\omega'(y)\supset\omega(y)$ so that $y\in\omega'(y)$ and $\chi(x,y)$ is continuous in $x\in\mathbb{R}^n$ and equals zero for $x\notin\bar{\omega}'(y)$, although such continuation is not actually used in the LBDIDEs.

2.3.3. *Globally smooth localization*

To simplify integral representation (19) even further, we eliminate the remaining (third) integral along $\Omega\cap\partial\omega(y)$, employing a cut-off function $\chi(x,y)$ smooth in $x\in\bar{\Omega}$ and vanishing on $\partial\omega(y)$ together with its normal derivative in x. Then, evidently, $\chi(x,y)$ is smooth in $x\in\mathbb{R}^n$ and the localized parametrix $P_\omega(\lambda;x,y)=\chi(x,y)P(\lambda;x,y)$ and its normal derivative vanish on $\partial\omega(y)$. For such a parametrix, both the third and fourth integrals disappear on the left-hand sides of LBDIDEs (20) and (22). Some examples of globally smooth cut-off functions localized on a ball or on a cube in $\mathbb{R}^n$ with a point y in its centre, are presented in [6]. Here we give also examples of globally smooth $\chi(x,y)$ localized on a polyhedron ω^p with p sides s_j, $j=1,2,...,p$. Let y be an internal point of the nonconcave polyhedron, Figure 2, or a boundary point of the polyhedron with the relaxed non-concavity described above, as in Figure 3. Then $\chi(x,y)$ can be taken in one of the following forms,

$$\chi(x,y)=\prod_{\bar{s}_j\not\ni y}\frac{\rho_j^2(x)}{\rho_j^2(y)},\quad x\in\bar{\omega}^p(y),\tag{24}$$

$$\chi(x,y)=\prod_{\bar{s}_j\not\ni y}\exp\left(1-\frac{\rho_j^2(y)}{\rho_j^2(x)}\right),\quad x\in\bar{\omega}^p(y),\tag{25}$$

where $\chi(x, y) = 0$ for $x \notin \bar{\omega}^p(y)$. Note that cut-off function (24) is continuous and has continuous first derivatives in $x \in \mathbb{R}^n$, while function (25) is infinitely smooth in $x \in \mathbb{R}^n$ for $y \in \omega^p(y)$.

3. Two-operator direct integro-differential formulations

In this section, we consider a more general quasi-linear PDE of the second order, whose coefficients depend not only on the unknown solution $u(x)$ but also on its gradient $\nabla u(x)$. In principle, one could apply the above direct (single-operator) approach of Section 2.2 to such equations and arrive at a direct quasi-linear BDIDE, which includes second derivatives of the unknown solution in the remainder R, (8), (10). To avoid this, we derive below a two-operator second Green identity combining the first Green identities of two different PDEs. This allows us to reduce the mixed BVP to a two-operator direct BDIDE with the first derivatives of the unknown solution at most.

3.1. Nonlinear "stationary potential compressible flow" problem and two-operator Green identities

Let us consider a mixed boundary-value problem for the following equation in a 2D or 3D open domain Ω,

$$[L(u)u](x) := \frac{\partial}{\partial x_k}\left[a(\nabla u(x), u(x), x)\frac{\partial u(x)}{\partial x_k}\right] = f(x), \quad x \in \Omega, \tag{26}$$

$$u(x) = \bar{u}(x), \quad x \in \partial_D\Omega, \tag{27}$$

$$[T(u)u](x) := a(\nabla u(x), u(x), x)\frac{\partial u(x)}{\partial x_k} = \bar{t}(x), \quad x \in \partial_N\Omega, \tag{28}$$

where $u(x)$ is unknown, $[L(\lambda)u](x) := \frac{\partial}{\partial x_k}\left[a(\nabla\lambda(x), \lambda(x), x)\frac{\partial u(x)}{\partial x_k}\right]$ is a linear differential operator, $[T(\lambda)u](x) := a(\nabla\lambda(x), \lambda(x), x)\partial u(x)/\partial n(x)$ is a linear surface-flux (or traction) operator and $a(\nabla\lambda(x), \lambda(x), x) > C > 0$ is a variable coefficient depending on a function $\lambda(x)$ and on its gradient $\nabla\lambda(x)$; $f(x)$ is a known right-hand side, $n(x)$ is an outward normal vector to the boundary $\partial\Omega$, $\bar{u}(x)$ and $\bar{t}(x)$ are known functions on the parts $\partial_D\Omega$ and $\partial_N\Omega$ of the boundary, respectively. The problem becomes a pure Neumann problem if $\partial_D\Omega = \emptyset$ and a pure Dirichlet problem if $\partial_N\Omega = \emptyset$. Such BVPs are encountered particularly in the stationary potential flow problem for a compressible fluid (for the stream function or the velocity potential $u(x)$) and in the static anti-plane problem of nonlinear elasticity for an inhomogeneous body (for the displacement $u(x)$).

The first Green identity for the differential operator $[L(u)u](x)$ has the form

$$\int_\Omega v(x)[L(u)u](x)\mathrm{d}\Omega(x) = \int_{\partial\Omega} v(x)[T(u)u](x)\mathrm{d}\Gamma(x) - \int_\Omega \frac{\partial v(x)}{\partial x_k}a(\nabla u(x), u(x), x)\frac{\partial u(x)}{\partial x_k}\mathrm{d}\Omega(x), \tag{29}$$

where $u(x)$ and $v(x)$ are arbitrary functions ensuring that the operators and integrals in (29) make sense.

Let us fix a point y and consider the linear differential operator with constant coefficients

$$[L^{(y)}(u)v](x) := \frac{\partial}{\partial x_k}\left[a(\nabla u(y), u(y), y)\frac{\partial v(x)}{\partial x_k}\right]$$

and write the first Green identity for the auxiliary operator $L^{(y)}(u)$,

$$\int_\Omega u(x)[L^{(y)}(u)v](x)\mathrm{d}\Omega(x) = \int_{\partial\Omega} u(x)[T^{(y)}(u)v](x)\mathrm{d}\Gamma(x) - \int_\Omega \frac{\partial u(x)}{\partial x_k} a(\nabla u(y), u(y), y)\frac{\partial v(x)}{\partial x_k}\mathrm{d}\Omega(x), \tag{30}$$

where $[T^{(y)}(u)v](x) := a(\nabla u(y), u(y), y)\partial v(x)/\partial n(x)$. Subtracting (29) from (30), we obtain the following two-operator second Green identity, *c.f.* (4),

$$\begin{aligned}\int_\Omega &\left\{u(x)[L^{(y)}(u)v](x) - v(x)[L(u)u](x)\right\}\mathrm{d}\Omega(x)\\ &= \int_{\partial\Omega}\left\{u(x)[T^{(y)}(u)v](x) - v(x)[T(u)u](x)\right\}\mathrm{d}\Gamma(x)\\ &+ \int_\Omega \frac{\partial v(x)}{\partial x_k}[a(\nabla u(x), u(x), x) - a(\nabla u(y), u(y), y)]\frac{\partial u(x)}{\partial x_k}\mathrm{d}\Omega(x).\end{aligned} \tag{31}$$

Note that if $L(u) = L^{(y)}(u)$, *i.e.*, $L(u)$ is a linear operator with constant coefficients; then, the last domain integral disappears, and the two-operator Green identity degenerates into its classical form (4).

3.2. Parametrix and quasi-linear two-operator direct integro-differential equations

Let $P^{(y)}(u; x, y)$ be a parametrix for the linear differential operator $[L^{(y)}(u)v](x)$ with constant coefficient associated with a point y, that is,

$$\begin{aligned}[L^{(y)}(u)P^{(y)}(u;\cdot,y)](x) &:= \frac{\partial}{\partial x_k}\left[a(\nabla u(y), u(y), y)\frac{\partial P^{(y)}(u;x,y)}{\partial x_k}\right]\\ &= \delta(x-y) + R^{(y)}(u;x,y),\end{aligned} \tag{32}$$

where the remainder term $R^{(y)}(u;x,y) = R(\nabla u(y), u(y), x, y)$ as function of $x \in \Omega$ has not more than a single weak (integrable) singularity.

If one chooses the fundamental solution $F^{(y)}(u;x,y)$ of the operator $L^{(y)}(u)$ as the parametrix, then $R^{(y)}(u;x,y) = 0$. Since $L^{(y)}(u)$ is a linear operator with constant (w.r.t. x) coefficients, its fundamental solution is readily available from the fundamental solution $F_\Delta(x,y)$ of the Laplace operator, $F^{(y)}(u;x,y) = F_\Delta(x,y)/a(\nabla u(y), u(y), y)$. Denoting $|x-y| = \sqrt{(x_k - y_k)(x_k - y_k)}$, we have

$$F^{(y)}(u;x,y) = \frac{\log|x-y|}{2\pi a(\nabla u(y), u(y), y)}, \quad x, y \in \mathbb{R}^2, \tag{33}$$

$$F^{(y)}(u;x,y) = \frac{-1}{4\pi a(\nabla u(y), u(y), y)|x-y|}, \quad x, y \in \mathbb{R}^3. \tag{34}$$

Assuming $u(x)$ is a solution of PDE (26) and using a parametrix $P^{(y)}(u;x,y)$ as $v(x)$ in the Green identity (31), one can obtain the following nonlinear two-operator third Green identity,

$$\begin{aligned}c(y)u(y) &- \int_{\partial\Omega} u(x)[T^{(y)}(u)P^{(y)}(u;\cdot,y)](x)\mathrm{d}\Gamma(x) + \int_{\partial\Omega} P^{(y)}(u;x,y)[T(u)u](x)\mathrm{d}\Gamma(x)\\ &- \int_\Omega \frac{\partial P^{(y)}(u;x,y)}{\partial x_k}[a(\nabla u(x), u(x), x) - a(\nabla u(y), u(y), y)]\frac{\partial u(x)}{\partial x_k}\mathrm{d}\Omega(x)\\ &+ \int_\Omega R^{(y)}(u;x,y)u(x)\mathrm{d}\Omega(x) = \int_\Omega P^{(y)}(u;x,y)f(x)\mathrm{d}\Omega(x),\end{aligned} \tag{35}$$

where $c(y)$ is given by (5). If the parametrix is a fundamental solution of the linear operator, $P^{(y)}(u;x,y)=F^{(y)}(u;x,y)$, then the last integral disappears on the left-hand side of (35). The penultimate domain integral stays nonetheless, and will disappear only if $L(u)=L^{(y)}(u)$, *i.e.*, if $L(u)$ is a linear operator with a constant coefficient. As follows *e.g.* from (33) and (34), the function $\partial P^{(y)}(u;x,y)/\partial x_k$ has generally a weak singularity at $x=y$. That makes the penultimate domain integral on the left-hand side of (35) weakly singular, and moreover, the order of the singularity is further reduced by up to one unit owing to the term $[a(\nabla u(x),u(x),x)-a(\nabla u(y),u(y),y)]$ if a and u are sufficiently smooth functions of their arguments.

3.2.1. *United formulation*

Using integral relation (35) we can now proceed as in Section 2.2. First, we substitute boundary conditions (27) and (28) in the integral terms of (35) and use (35) at $y\in\bar{\Omega}$,

$$\begin{aligned} &c(y)u(y)-\int_{\partial_N\Omega}u(x)[T^{(y)}(u)P^{(y)}(u;\cdot,y)](x)\mathrm{d}\Gamma(x)+\int_{\partial_D\Omega}P^{(y)}(u;x,y)[T(u)u](x)\mathrm{d}\Gamma(x)\\ &\quad-\int_\Omega\frac{\partial P^{(y)}(u;x,y)}{\partial x_k}[a(\nabla u(x),u(x),x)-a(\nabla u(y),u(y),y)]\frac{\partial u(x)}{\partial x_k}\mathrm{d}\Omega(x)\\ &\quad+\int_\Omega R^{(y)}(u;x,y)u(x)\mathrm{d}\Omega(x)=\mathcal{F}(u;y),\quad y\in\bar{\Omega}, \end{aligned} \tag{36}$$

$$\begin{aligned} \mathcal{F}(u;y):=&\int_{\partial_D\Omega}\bar{u}(x)[T^{(y)}(u)P^{(y)}(u;\cdot,y)](x)\mathrm{d}\Gamma(x)-\int_{\partial_N\Omega}P^{(y)}(u;x,y)\bar{t}(x)\mathrm{d}\Gamma(x)\\ &+\int_\Omega P^{(y)}(u;x,y)f(x)\mathrm{d}\Omega(x). \end{aligned} \tag{37}$$

The second-kind form of BDIDE (36) looks attractive for constructing iterative solution algorithms.

3.2.2. *Partly segregated formulation*

On the other hand, substituting $\bar{u}(y)$ also for the out-of-integral term $u(y)$ at $y\in\partial\Omega_D$ and introducing a new variable $t(x)=[T(u)u](x)$ for the unknown flux at $x\in\partial\Omega_D$ in (36), we may reduce BVP (26–28) to the following partly segregated quasi-linear two-operator direct LBDIDE for $u(x)$ at $x\in\Omega\cup\partial_N\Omega$ and $t(x)$ at $x\in\partial_D\Omega$,

$$\begin{aligned} &c^0(y)u(y)-\int_{\partial_N\Omega}u(x)[T^{(y)}(u)P^{(y)}(u;\cdot,y)](x)\mathrm{d}\Gamma(x)+\int_{\partial_D\Omega}P^{(y)}(u;x,y)[T(u)u](x)\mathrm{d}\Gamma(x)\\ &\quad-\int_\Omega\frac{\partial P^{(y)}(u;x,y)}{\partial x_k}[a(\nabla u(x),u(x),x)-a(\nabla u(y),u(y),y)]\frac{\partial u(x)}{\partial x_k}\mathrm{d}\Omega(x)\\ &\quad+\int_\Omega R^{(y)}(u;x,y)u(x)\mathrm{d}\Omega(x)=\mathcal{F}^0(u;y),\quad y\in\bar{\Omega}, \end{aligned} \tag{38}$$

$$\mathcal{F}^0(u;y):=[c^0(y)-c(y)]\bar{u}(y)+\mathcal{F}(u;y), \tag{39}$$

where c^0 is given by (15).

Note that BDIDEs (36) and (38) involve at most the first derivatives of the unknown solution $u(x)$ through the coefficient $a(\nabla u,u,\cdot)$ both directly in the third (domain) integral term on the left-hand side and in the operators $T(u)$, $T^{(y)}(u)$, and in the functions $P^{(y)}(u;x,y)$ and $R^{(y)}(u;x,y)$. Note also that not only the left-hand sides of BDEDEs (36) and (38) but also their right-hand sides $\mathcal{F}$ and $\mathcal{F}^0$ do depend on the unknown solution u. If the original BVP (26–28) is linear, *i.e.*, the coefficient a is independent of u and ∇u, then T, $T^{(y)}$, $P^{(y)}$,

$R^{(y)}$, $\mathcal{F}$ and $\mathcal{F}^0$ do not depend on u and ∇u either, and BDEDEs (36) and (38) degenerate into linear BDEDEs with the known right-hand sides $\mathcal{F}$ and $\mathcal{F}^0$.

3.3. Localized parametrices and quasi-linear two-operator direct BDIDE

Each of BDIDEs (36) and (38) can be reduced after some discretization to a system of nonlinear algebraic equations that can be solved numerically. The system will include unknowns not only at the boundary but also at internal points. Moreover, since the commonly used parametrices, *e.g.*, fundamental solutions (33), (34), are highly nonlocal, the matrix of the system will be fully populated and this makes its numerical solution more expensive. See for example [13, 14], where some indirect BDIEs for *linear* elastic-shell problems with variable coefficients were analysed and solved numerically. To avoid this difficulty, one can construct *localized* parametrices and consequently *localized* boundary-domain integro-differential equations (LBDIDEs).

Thus, as in Section 2.3, we can consider a function $P_\omega^{(y)}(u; x, y) = \chi(x, y)P^{(y)}(u; x, y)$, where $P^{(y)}(u)$ is an available (not localized) parametrix to the linear operator $L^{(y)}(u)$, *e.g.*, its fundamental solution $F^{(y)}(u; x, y)$, and $\chi(x, y)$ is a cut-off function, such that $\chi(y, y) = 1$ and $\chi(x, y) = 0$ at x not belonging to the closure of an open localization domain $\omega(y)$ (a vicinity of y); see Figure 1. Then, similar to the reasoning in Section 2.3, $P_\omega^{(y)}(u; x, y)$ is the localized parametrix of the linear operator $L^{(y)}(u)$, at least if χ is sufficiently smooth, and the localized remainder is

$$\begin{aligned} R_\omega^{(y)}(u; x, y) &= R^{(y)}(u; x, y) - [L^{(y)}(u)\{(1 - \chi(\cdot, y))P^{(y)}(u; \cdot, y)\}](x) \\ &= \chi(x, y)R^{(y)}(u; x, y) + P^{(y)}(u; x, y)[L^{(y)}(u)\chi(\cdot, y)](x) \\ &\quad + 2\frac{\partial \chi(x, y)}{\partial x_i} a(\nabla u(y), u(y), y)\frac{\partial P^{(y)}(u; x, y)}{\partial x_i}. \end{aligned} \tag{40}$$

Note that if $P^{(y)}(u; x, y)$ is a fundamental solution of the operator $L^{(y)}(u)$, then $R^{(y)}(u; x, y) = 0$ but generally $R_\omega^{(y)}(u; x, y) \neq 0$.

3.3.1. *Discontinuous localization*

Suppose $\chi(x, y)$ is smooth in $x \in \bar\omega(y)$ but not necessarily zero at $x \in \partial\omega(y)$, as represented by (16). Then $P_\omega^{(y)}(u; x, y)$ is a discontinuous localized parametrix at $x \in \bar\Omega$ and $P_\omega^{(y)}(u; x, y) = R_\omega^{(y)}(u; x, y) = 0$ if $x \notin \bar\omega(y)$. Assume that y lies either inside the domain $\omega(y)$ or on the coinciding part of the localization and global domain boundaries, $\partial\omega(y) \cap \partial\Omega$, such that $\alpha(y; \Omega) = \alpha(y; \omega(y))$. Substituting $P_\omega^{(y)}(u; x, y)$ for $P^{(y)}(u; x, y)$ in (35), where $\bar\Omega$ is replaced by the intersection $\bar\omega(y) \cap \bar\Omega$, and taking $u(x)$ as a solution to (26), we arrive at the nonlinear two-operator third Green identity localized on $\bar\omega(y) \cap \bar\Omega$,

$$\begin{aligned} c(y)u(y) &- \int_{\bar\omega(y)\cap\partial\Omega} u(x)[T^{(y)}(u)P_\omega^{(y)}(u; \cdot, y)](x)\mathrm{d}\Gamma(x) + \int_{\bar\omega(y)\cap\partial\Omega} P_\omega^{(y)}(u; x, y)[T(u)u](x)\mathrm{d}\Gamma(x) \\ &- \int_{\Omega\cap\partial\omega(y)} u(x)[T^{(y)}(u)P_\omega^{(y)}(u; \cdot, y)](x)\mathrm{d}\Gamma(x) + \int_{\Omega\cap\partial\omega(y)} P_\omega^{(y)}(u; x, y)[T(u)u](x)\mathrm{d}\Gamma(x) \\ &- \int_{\omega(y)\cap\Omega} \frac{\partial P_\omega^{(y)}(u; x, y)}{\partial x_k}[a(\nabla u(x), u(x), x) - a(\nabla u(y), u(y), y)]\frac{\partial u(x)}{\partial x_k}\mathrm{d}\Omega(x) \\ &+ \int_{\omega(y)\cap\Omega} R_\omega^{(y)}(u; x, y)u(x)\mathrm{d}\Omega(x) = \int_{\omega(y)\cap\Omega} P_\omega^{(y)}(u; x, y)f(x)\mathrm{d}\Omega(x), \quad y \in \mathbb{R}^n, \end{aligned} \tag{41}$$

where $c(y) = c(y; \Omega)$ is given by (5).

Note that the last integral on the left-hand side of (41) disappears if $\chi(x, y)$ is a piecewise constant function (18) and the parametrix before the localization is a fundamental solution, $P^{(y)}(u; x, y) = F^{(y)}(u; x, y)$.

United formulation. We can now substitute boundary conditions (27) and (28) in the first two integrals of Green's third two-operator identity (41), leave $T(u)$ as the differential operator acting on u, at $\partial_D\Omega$, and use the following LBDIDE at $y \in \bar{\Omega}$,

$$\begin{aligned}
c(y)u(y) &- \int_{\bar{\omega}(y)\cap\partial_N\Omega} u(x)[T^{(y)}(u)P_\omega^{(y)}(u; \cdot, y)](x)\mathrm{d}\Gamma(x) \\
&+ \int_{\bar{\omega}(y)\cap\partial_D\Omega} P_\omega^{(y)}(u; x, y)[T(u)u](x)\mathrm{d}\Gamma(x) \\
&- \int_{\Omega\cap\partial\omega(y)} u(x)[T^{(y)}(u)P_\omega^{(y)}(u; \cdot, y)](x)\mathrm{d}\Gamma(x) \\
&+ \int_{\Omega\cap\partial\omega(y)} P_\omega^{(y)}(u; x, y)[T(u)u](x)\mathrm{d}\Gamma(x) \\
&- \int_{\omega(y)\cap\Omega} \frac{\partial P_\omega^{(y)}(u; x, y)}{\partial x_k}[a(\nabla u(x), u(x), x) - a(\nabla u(y), u(y), y)]\frac{\partial u(x)}{\partial x_k}\mathrm{d}\Omega(x) \\
&+ \int_{\omega(y)\cap\Omega} R_\omega^{(y)}(u; x, y)u(x)\mathrm{d}\Omega(x) = \mathcal{F}_\omega(u; y), \quad y \in \bar{\Omega},
\end{aligned} \tag{42}$$

$$\begin{aligned}
\mathcal{F}_\omega(u; y) := &\int_{\bar{\omega}(y)\cap\partial_D\Omega} \bar{u}(x)[T^{(y)}(u)P_\omega^{(y)}(u; \cdot, y)](x)\mathrm{d}\Gamma(x) \\
&- \int_{\bar{\omega}(y)\cap\partial_N\Omega} P_\omega^{(y)}(u; x, y)\bar{t}(x)\mathrm{d}\Gamma(x) + \int_{\omega(y)\cap\Omega} P_\omega^{(y)}(u; x, y)f(x)\mathrm{d}\Omega(x).
\end{aligned} \tag{43}$$

Partly segregated formulation. On the other hand, substitution of $\bar{u}(y)$ also for the out-of-integral term $u(y)$ at $y \in \partial\Omega_D$ and introduction of a new variable $t(x) = [T(u)u](x)$ for the unknown flux at $x \in \partial\Omega_D$ in (42) reduce BVP (26–28) to the following partly segregated quasi-linear two-operator direct LBDIDE for $u(x)$ at $x \in \Omega \cup \partial_N\Omega$ and $t(x)$ at $x \in \partial_D\Omega$,

$$\begin{aligned}
c^0(y)u(y) &- \int_{\bar{\omega}(y)\cap\partial_N\Omega} u(x)[T^{(y)}(u)P_\omega^{(y)}(u; \cdot, y)](x)\mathrm{d}\Gamma(x) \\
&+ \int_{\bar{\omega}(y)\cap\partial_D\Omega} P_\omega^{(y)}(u; x, y)t(x)\mathrm{d}\Gamma(x) \\
&- \int_{\Omega\cap\partial\omega(y)} u(x)[T^{(y)}(u)P_\omega^{(y)}(u; \cdot, y)](x)\mathrm{d}\Gamma(x) \\
&+ \int_{\Omega\cap\partial\omega(y)} P_\omega^{(y)}(u; x, y)[T(u)u](x)\mathrm{d}\Gamma(x) \\
&- \int_{\omega(y)\cap\Omega} \frac{\partial P_\omega^{(y)}(u; x, y)}{\partial x_k}[a(\nabla u(x), u(x), x) - a(\nabla u(y), u(y), y)]\frac{\partial u(x)}{\partial x_k}\mathrm{d}\Omega(x) \\
&+ \int_{\omega(y)\cap\Omega} R_\omega^{(y)}(u; x, y)u(x)\mathrm{d}\Omega(x) = \mathcal{F}_\omega^0(u; y), \qquad y \in \bar{\Omega},
\end{aligned} \tag{44}$$

$$\mathcal{F}_\omega^0(u; y) := [c^0(y) - c(y)]\bar{u}(y) + \mathcal{F}_\omega(u; y). \tag{45}$$

3.3.2. *Continuous localizations*

To eliminate the integrals involving $T(u)u$ on $\Omega\cap\partial\omega(y)$, that is the fourth integrals on the left-hand sides of (41), (42) and (44), one can construct a localized parametrix $P_\omega^{(y)}(u;x,y)$ that vanishes on the boundary part $\partial\omega(y)$ (except perhaps a neighbourhood of $y\in\partial\omega(y)\cap\partial\Omega$) but not necessarily with vanishing parametrix flux $[T^{(y)}(u)P_\omega^{(y)}(u;\cdot,y)](x)$. As described in Section 2.3.2, this may be done by choosing $P_\omega^{(y)}(u;x,y)$ as a Green function for $\omega(y)$ if $\omega(y)$ is a ball. A more general way is to use an appropriate cut-off function $\chi(x,y)$; some examples of such cut-off functions are given in Section 2.3.2.

3.3.3. *Globally smooth localization*

To simplify the BDIDEs even further by eliminating the remaining (third) integral along $\partial\omega(y)$, one can employ a globally smooth cut-off function $\chi(x,y)$, which vanishes on $\partial\omega(y)$ together with its normal derivative in x (except possibly a neighbourhood of $y\in\partial\omega(y)\cap\partial\Omega$). Then the same holds true also for the parametrix $P_\omega^{(y)}(u;x,y)=\chi(x,y)P^{(y)}(u;x,y)$. For such a parametrix, the third and fourth integrals disappear on the left-hand side of (41), (42) and (44). Some examples of globally smooth cut-off functions are presented in Section 2.3.3.

4. Discretization of quasi-linear LBDIDEs

To reduce any of the quasi-linear LBDIDEs obtained above to a sparsely populated system of quasi-linear algebraic equations, *e.g.*, by a collocation method, one has to employ a local interpolation or approximation formula for the unknown function $u(x)$. As has been demonstrated, there is a lot of flexibility in constructing appropriate cut-off functions. We will consider the general case of the discontinuous localization and show the simplifications for more smooth localizations.

4.1. Mesh-based discretization

4.1.1. *Mesh-based interpolation*

Suppose the domain Ω is covered by a mesh of closures of disjoint open domain elements e_k with nodes set up at the corners, edges, faces, or inside the elements. Let J be the total number of nodes x^i $(i=1,2,...,J)$. One can use each node x^i as a collocation point for an LBDIDE with a localization domain $\omega(x^i)$. Let the union of closures of the domain elements that intersect $\omega(x^i)$ be called the *total* localization domain $\tilde\omega^i$; Figure 4. Then the closure $\bar\omega(x^i)\cap\bar\Omega$ belongs to $\tilde\omega^i$. If $\omega(x^i)$ is sufficiently small, then $\tilde\omega^i$ consists only of the elements adjacent to the collocation point x^i. If $\omega(x^i)$ is chosen *ab initio* as consisting only of the elements adjacent to the collocation point x^i, then $\tilde\omega^i=\bar\omega(x^i)$. Let $u\{\tilde\omega^i\}$ be the array of the function values $u(x^j)$ at the node points $x^j\in\tilde\omega^i$, and $J_{\tilde\omega^i}$ be the number of these node points.

Let $u(x)=\sum_j u(x^j)\phi_{kj}(x)$ be a continuous piece-wise smooth interpolation of $u(x)$ at any point $x\in\Omega$ along the values $u(x^j)$ at the node points x^j belonging to the same element $\bar e_k\subset\Omega$ as x, and the shape functions $\phi_{kj}(x)$ be localized on $\bar e_k$. Collecting the interpolation formulae for all $x\in\tilde\omega^i$, we have

$$u(x)=\sum_{x^j\in\tilde\omega^i}u(x^j)\Phi_j(x),\quad \Phi_j(x)=\begin{cases}\phi_{kj}(x) & \text{if } x,x^j\in\bar e_k,\\ 0 & \text{otherwise,}\end{cases}\tag{46}$$

$$\nabla u(x)=\sum_{x^j\in\tilde\omega^i}u(x^j)\nabla\Phi_j(x),\ \nabla\Phi_j(x)=\begin{cases}\nabla\phi_{kj}(x) & \text{if } x,x^j\in\bar e_k,\\ 0 & \text{otherwise.}\end{cases}\tag{47}$$

Consequently, $\Phi_j(x)=\nabla\Phi_j(x)=0$ if $x\in\bar\omega^i$ but $x^j\notin\tilde\omega^i$.

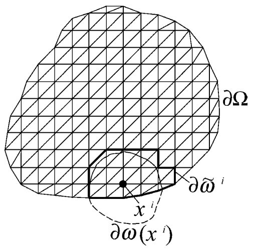

Figure 4. A localization domain $\omega(x^i)$ and a total localization domain $\tilde{\omega}^i$ associated with a collocation point x^i of a body Ω at a mesh-based discretization.

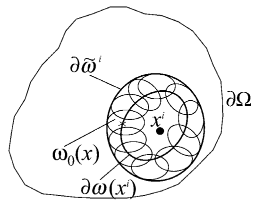

Figure 5. A localization domain $\omega(x^i)$ and a total localization domain $\tilde{\omega}^i$ associated with a collocation point x^i of a body Ω for a mesh-less discretization.

Since interpolation (46) is piece-wise smooth, expressions (47) deliver non-unique values for $\nabla u(x)$ on the element interfaces and particularly at the apexes x^i of different adjoint elements e_k. This brings no complications for direct BDIDEs (20) or (22) of BVP (1–3) since the solution gradients appear either in the domain integrals or in the boundary integrals with the gradients taken from the corresponding side of the boundary. On the other hand, for two-operator direct BDIDEs (42) or (44) of BVP (26–28) one has to estimate $\nabla u(y)$ to calculate the coefficient $a(\nabla u(y), u(y), y)$ and, consequently $T^{(y)}(u)$, $P^{(y)}(u; x, y)$ and $R^{(y)}(u; x, y)$ at $y = x^i$. A possible way out is to assign

$$\nabla u(x^i) := \sum_{\bar{e}_k \ni x^i} \frac{\alpha_k(x^i)}{\alpha(x^i)} \nabla u^k(x^i), \quad \nabla u^k(x^i) := \sum_{x^j \in \bar{e}_k} u(x^j) \nabla \phi_{kj}(x^i), \tag{48}$$

where $\alpha_k(x^i)$ is an interior space angle at the apex x^i of the element e_k and $\alpha(x^i) = \sum_{\bar{e}_k \ni x^i} \alpha_k(x^i)$.

We can also use a local interpolation of the unknown flux variable $t(x)$ along only boundary nodes belonging to $\tilde{\omega}^i \cap \partial_D \Omega$,

$$t(x) = \sum_{x^j \in \tilde{\omega}^i \cap \partial_D \Omega} t(x^j) \Phi'_j(x), \quad x \in \tilde{\omega}^i \cap \partial_D \Omega. \tag{49}$$

Here $\Phi'_j(x)$ are the shape functions on the boundary obtained similar to $\Phi_j(x)$ in (46).

4.1.2. *Mesh-based discretization of quasi-linear direct LBDIDEs*

Partly segregated formulation. After substituting the above interpolations in LBDIDE (22) of the direct partly segregated formulation at the collocation points $y = x^i \in \bar{\Omega}$, and taking into account (2), we derive the following system of J quasi-linear algebraic equations for J unknowns: $u(x^j)$, $x^j \in \Omega \cup \partial_N \Omega$ and $t(x^j)$, $x^j \in \partial_D \Omega$,

$$\begin{aligned} c^0(x^i) u(x^i) &+ \sum_{x^j \in \tilde{\omega}^i \setminus \partial_D \Omega} K^0_{ij}(u\{\tilde{\omega}^i\}) u(x^j) + \sum_{x^j \in \partial_D \Omega \cap \tilde{\omega}^i} Q_{ij}(u\{\tilde{\omega}^i\}) t(x^j) \\ &= \mathcal{F}^0_\omega(u\{\tilde{\omega}^i\}, x^i) - \sum_{x^j \in \partial_D \Omega \cap \tilde{\omega}^i} K^0_{ij}(u\{\tilde{\omega}^i\}) \bar{u}(x^j), \qquad x^i \in \bar{\Omega}, \end{aligned} \tag{50}$$

$$K^0_{ij}(u\{\tilde{\omega}^i\}) = -\int_{\bar{\omega}(x^i)\cap\partial_N\Omega} \Phi_j(x)[T(u\{\tilde{\omega}^i\})P_\omega(u\{\tilde{\omega}^i\};\cdot,x^i)](x)\mathrm{d}\Gamma(x)$$
$$-\int_{\Omega\cap\partial\omega(x^i)} \Phi_j(x)[T(u\{\tilde{\omega}^i\})P_\omega(u\{\tilde{\omega}^i\};\cdot,x^i)](x)\mathrm{d}\Gamma(x)$$
$$+\int_{\Omega\cap\partial\omega(x^i)} P_\omega(u\{\tilde{\omega}^i\};x,x^i)[T(u\{\tilde{\omega}^i\})\Phi_j](x)\mathrm{d}\Gamma(x)$$
$$+\int_{\omega(x^i)\cap\Omega} R_\omega(u\{\tilde{\omega}^i\};x,x^i)\Phi_j(x)\mathrm{d}\Omega(x), \tag{51}$$
$$Q_{ij}(u\{\tilde{\omega}^i\}) = \int_{\bar{\omega}(x^i)\cap\partial_D\Omega} P_\omega(u\{\tilde{\omega}^i\};x,x^i)\Phi'_j(x)\mathrm{d}\Gamma(x) \tag{52}$$

(no sum in i).

United formulation. Instead, one can derive another system of J quasi-linear algebraic equations for J unknowns $u(x^j)$, $x^j\in\bar{\Omega}$, if one substitutes interpolation formulae (46) in LBDIDE (20) of the direct united formulation,

$$c(x^i)u(x^i) + \sum_{x^j\in\tilde{\omega}^i} K_{ij}(u\{\tilde{\omega}^i\})u(x^j) = \mathcal{F}_\omega(u\{\tilde{\omega}^i\},x^i), \quad x^i\in\bar{\Omega}, \tag{53}$$

$$K_{ij}(u\{\tilde{\omega}^i\}) = K^0_{ij}(u\{\tilde{\omega}^i\}) + \int_{\bar{\omega}(x^i)\cap\partial_D\Omega} P_\omega(x,x^i)[T(u)\Phi_j](x)\mathrm{d}\Gamma(x) \tag{54}$$

no sum in i, and K^0_{ij} is given by (51).

The approximate flux operator $T(u\{\tilde{\omega}^i\})$, localized parametrix $P_\omega(u\{\tilde{\omega}^i\};x,x^i)$ and localized remainder $R_\omega(u\{\tilde{\omega}^i\};x,x^i)$ in (51), (52) and (54) are expressed in terms of the set of unknowns $u\{\tilde{\omega}^i\}:=\{u(x^j),\ x^j\in\tilde{\omega}^i\}$. The expressions are obtained after substituting interpolation formulae (46), (47) for u in the coefficient $a(u;\cdot)$ in the definitions for $T(u)$, $P_\omega(u;x,y)$ and $R_\omega(u;x,y)$ in Section 2. The right-hand side components $\mathcal{F}_\omega(u\{\tilde{\omega}^i\},x^i)$ and $\mathcal{F}^0_\omega(u\{\tilde{\omega}^i\},x^i)$ are obtained after using interpolation formulae (46), (47) for u in (21) and (23). The relations $u(x^j)=\bar{u}(x^j)$, $x^j\in\partial_D\Omega$, should be also employed while interpolating u in the partly segregated formulation (50–52). Instead, $u(x^j)$, $x^j\in\partial_D\Omega$, should be considered as unknown while interpolating u in the united formulation (53), (54), (51).

Thus, the algebraic systems (52) and (53) are nonlinear (we call them quasi-linear since "freezing" the unknown solution in the matrices of coefficients and right-hand sides leads to linear systems).

Note that if the cut-off function $\chi(x,x^i)$ and its normal derivative are equal zero at x on the boundary $\partial\omega(x^i)$, then the second and third integrals (along $\Omega\cap\partial\omega(x^i)$) disappear on the right-hand side of (51).

4.1.3. *Mesh-based discretization of quasi-linear two-operator direct LBDIDEs*

Partly segregated formulation. After substituting interpolations (46–49) in LBDIDE (44) of the two-operator direct partly segregated formulation and taking into account (27), we derive a system of J quasi-linear algebraic equations for J unknowns: $u(x^j)$, $x^j\in\Omega\cup\partial_N\Omega$ and $t(x^j)$, $x^j\in\partial_D\Omega$. The system has a form similar to (50),

$$c^0(x^i)u(x^i) + \sum_{x^j\in\tilde{\omega}^i\setminus\partial_D\Omega} K^0_{ij}(u\{\tilde{\omega}^i\})u(x^j) + \sum_{x^j\in\partial_D\Omega\cap\tilde{\omega}^i} Q_{ij}(u\{\tilde{\omega}^i\})t(x^j)$$
$$= \mathcal{F}^0_\omega(u\{\tilde{\omega}^i\},x^i) - \sum_{x^j\in\partial_D\Omega\cap\tilde{\omega}^i} K^0_{ij}(u\{\tilde{\omega}^i\})\bar{u}(x^j), \quad x^i\in\bar{\Omega} \tag{55}$$

(no sum in i). Here, however,

$$\begin{aligned}K^0_{ij}(u\{\tilde{\omega}^i\}) = &-\int_{\bar{\omega}(x^i)\cap\partial_N\Omega} \Phi_j(x)[T^{(x^i)}(u\{\tilde{\omega}^i\})P^{(x^i)}_{\omega}(u\{\tilde{\omega}^i\};\cdot,x^i)](x)\mathrm{d}\Gamma(x)\\ &-\int_{\Omega\cap\partial\omega(x^i)} \Phi_j(x)[T^{(x^i)}(u\{\tilde{\omega}^i\})P^{(x^i)}_{\omega}(u\{\tilde{\omega}^i\};\cdot,x^i)](x)\mathrm{d}\Gamma(x)\\ &+\int_{\Omega\cap\partial\omega(x^i)} P^{(x^i)}_{\omega}(u\{\tilde{\omega}^i\};x,x^i)[T(u\{\tilde{\omega}^i\})\Phi_j](x)\mathrm{d}\Gamma(x)\\ &-\int_{\omega(x^i)\cap\Omega} \frac{\partial P^{(x^i)}_{\omega}(u\{\tilde{\omega}^i\};x,x^i)}{\partial x_k}[a(\nabla u(x),u(x),x)\\ &-a(\nabla u(x^i),u(x^i),x^i)]\frac{\partial\Phi_j(x)}{\partial x_k}\mathrm{d}\Omega(x)\\ &+\int_{\omega(x^i)\cap\Omega} R^{(x^i)}_{\omega}(u\{\tilde{\omega}^i\};x,x^i)\Phi_j(x)\mathrm{d}\Omega(x), \end{aligned}\tag{56}$$

$$Q_{ij}(u\{\tilde{\omega}^i\}) = \int_{\bar{\omega}(x^i)\cap\partial_D\Omega} P^{(x^i)}_{\omega}(u\{\tilde{\omega}^i\};x,x^i)\Phi'_j(x)\mathrm{d}\Gamma(x), \tag{57}$$

(no sum in i).

United formulation. Instead, one can arrive at another system of J quasi-linear algebraic equations for the J unknowns $u(x^j)$, $x^j\in\bar{\Omega}$, if one substitutes the interpolation formulae (46–48) in LBDIDE (42) of the two-operator direct united formulation

$$c(x^i)u(x^i) + \sum_{x^j\in\tilde{\omega}^i} K_{ij}(u\{\tilde{\omega}^i\})u(x^j) = \mathcal{F}_{\omega}(u\{\tilde{\omega}^i\};x^i), \quad x^i\in\bar{\Omega}, \tag{58}$$

$$K_{ij}(u\{\tilde{\omega}^i\}) = K^0_{ij}(u\{\tilde{\omega}^i\}) + \int_{\bar{\omega}(x^i)\cap\partial_D\Omega} P^{(x^i)}_{\omega}(u\{\tilde{\omega}^i\};x,x^i)[T(u\{\tilde{\omega}^i\})\Phi_j](x)\mathrm{d}\Gamma(x), \tag{59}$$

no sum in i, and K^0_{ij} is given by (56).

The approximate flux operators $T(u\{\tilde{\omega}^i\})$ and $T^{(x^i)}(u\{\tilde{\omega}^i\})$, localized parametrix $P^{(x^i)}_{\omega}(u\{\tilde{\omega}^i\};x,x^i)$ and localized remainder $R^{(x^i)}_{\omega}(u\{\tilde{\omega}^i\};x,x^i)$ in (56), (57) and (59) are expressed in terms of the set of unknowns $u\{\tilde{\omega}^i\} := \{u(x^j),\ x^j\in\tilde{\omega}^i\}$. The expressions are obtained after substituting interpolation formulae (46), (47) for u in the coefficient $a(u;\cdot)$ in the definitions for $T(u)$, $T^{(y)}(u)$, $P^{(y)}_{\omega}(u;x,y)$ and $R^{(y)}_{\omega}(u;x,y)$ in Section 3. The right-hand side components $\mathcal{F}_{\omega}(u\{\tilde{\omega}^i\},x^i)$ and $\mathcal{F}^0_{\omega}(u\{\tilde{\omega}^i\},x^i)$ in (58) and (55) are obtained after similarly using interpolation formulae (46), (47) for u in (43) and (45). The relations $u(x^j)=\bar{u}(x^j)$, $x^j\in\partial_D\Omega$, should be also employed while interpolating u in the partly segregated formulation (55–57). Instead, $u(x^j)$, $x^j\in\partial_D\Omega$, should be considered as unknown while interpolating u in the united formulation (58), (59), (56).

Note that the last integral terms (with $R^{(x^i)}_{\omega}$) disappear on the right-hand side of (56) if the parametrix $P^{(x^i)}_{\omega}(x,x^i)$ is the fundamental solution $F^{(x^i)}(x,x^i)$ (which implies $\chi(x,x^i)=\{1$ if $x\in\omega(x^i)$, 0 if $x\notin\omega(x^i)\}$). On the other hand, if the cut-off function $\chi(x,x^i)$ and its normal derivative are equal to zero at x on the boundary $\partial\omega(x^i)$, then the second and third integrals (along $\Omega\cap\partial\omega(x^i)$) disappear on the right-hand side of (56).

4.2. Mesh-less discretization

4.2.1. *Mesh-less approximation*

For a mesh-less discretization, one needs a method of local interpolation or approximation of a function along randomly distributed nodes x^i. We will suppose that all the approximation nodes x^i belong to $\bar{\Omega}$, and will use them also as collocation points for the LBDIDEs discretization. As before, let J be the total number of nodes x^j $(i=1,2,...,J)$, including J_D nodes on $\partial_D\Omega$. Let us consider a mesh-less method, for example, the moving least squares (MLS) method (see *e.g.* [15], [11, 12] and the references therein), that leads to the following approximation of a function $u(x)$,

$$u(x)=\sum_{x^j\in\omega_0(x)}\hat{u}(x^j)\Phi_j(x),\quad x\in\Omega. \tag{60}$$

Here $\Phi_j(x)$ are known smooth shape functions such that $\Phi_j(x)=0$ if $x^j\notin\omega_0(x)$, $\omega_0(x)$ is a localization domain of the approximation formula, and $\hat{u}(x^j)$ are unknown values of an auxiliary function $\hat{u}(x)$ at the nodes x^j, that is, the so-called δ-property is not assumed for approximation (60).

Let $\omega(x^i)$ be a localization domain around a node x^i. Then, for all $x\in\bar{\omega}(x^i)$, the total approximation of $u(x)$ can be written in the following local form,

$$u(x)=\sum_{x^j\in\tilde{\omega}^i}\hat{u}(x^j)\Phi_j(x),\quad \nabla u(x)=\sum_{x^j\in\tilde{\omega}^i}\hat{u}(x^j)\nabla\Phi_j(x),\qquad x\in\bar{\omega}(x^i), \tag{61}$$

where $\tilde{\omega}^i:=\cup_{x\in\bar{\omega}(x^i)\cap\bar{\Omega}}\omega_0(x)$ is a total localization domain; Figure 5. Consequently, $\Phi_j(x)=\nabla\Phi_j(x)=0$ if $x\in\bar{\omega}(x^i)$ and $x^j\notin\tilde{\omega}^i$. Let $J_{\tilde{\omega}^i}$ be the number of nodes $x^j\in\tilde{\omega}^i$ and $\hat{u}\{\tilde{\omega}^i\}$ be the array of the function values $\hat{u}(x^j)$ at the node points $x^j\in\tilde{\omega}^i$. Since our approximation (61) for u is smooth, its gradient approximation in (61) is continuous, and we do not need special formulae like (48) for calculating gradients $\nabla u(x^{(i)})$ at the collocation points $x^{(i)}$.

We can also use a local approximation of $t(x)$ along only boundary nodes belonging to $\tilde{\omega}^i\cap\partial_D\Omega$,

$$t(x)=\sum_{x^j\in\tilde{\omega}^i\cap\partial_D\Omega}\hat{t}(x^j)\Phi'_j(x),\qquad x\in\bar{\omega}(x^i)\cap\partial_D\Omega. \tag{62}$$

Here $\Phi'_j(x)$ are the shape functions on the boundary, obtained similarly to $\Phi_j(x)$ in (61).

4.2.2. *Mesh-less discretization of quasi-linear direct LBDIDEs*

Partly segregated formulation. After substituting approximations (61), (62) in LBDIDE (22) and boundary condition (2), we arrive at the following system of $J+J_D$ quasi-linear algebraic equations with respect to the J unknowns $\hat{u}(x^j)$, $x_j\in\bar{\Omega}$, and J_D unknowns $\hat{t}(x^j)$, $x_j\in\partial_D\Omega$,

$$\sum_{x^j\in\tilde{\omega}^i}\left[c^0(x^i)\Phi_j(x^i)+K^0_{ij}(\hat{u}\{\tilde{\omega}^i\})\right]\hat{u}(x^j)+\sum_{x^j\in\partial_D\Omega\cap\tilde{\omega}^i}Q_{ij}(\hat{u}\{\tilde{\omega}^i\})\hat{t}(x^j)$$
$$=\mathcal{F}^0_\omega(\hat{u}\{\tilde{\omega}^i\},x^i),\qquad x^i\in\bar{\Omega}, \tag{63}$$

$$\sum_{x^j\in\tilde{\omega}^i}\hat{u}(x^j)\Phi_j(x^i)=\bar{u}(x^i),\qquad x^i\in\partial_D\Omega,\ \text{no sum in } i. \tag{64}$$

United formulation. Alternatively, one can derive another quasi-linear system of J algebraic equations with respect to the J unknowns $\hat{u}(x^j)$, $x^j\in\bar{\Omega}$, if one substitutes approximation formulae (61) in LBDIDE (20),

$$\sum_{x^j\in\tilde{\omega}^i}\left[c(x^i)\Phi_j(x^i)+K_{ij}(\hat{u}\{\tilde{\omega}^i\})\right]\hat{u}(x^j)=\mathcal{F}_\omega(\hat{u}\{\tilde{\omega}^i\},x^i),\qquad x^i\in\bar{\Omega},\qquad \text{no sum in } i. \tag{65}$$

The matrices K^0_{ij}, Q_{ij}, K_{ij} in (63) and (65) are given by expressions (51), (52) and (54) with the shape functions Φ_j, Φ'_j from (61) and (62) and $u\{\tilde{\omega}^i\}$ replaced by $\hat{u}\{\tilde{\omega}^i\}$. Expressions for $T(\hat{u}\{\tilde{\omega}^i\})$, $P_\omega(\hat{u}\{\tilde{\omega}^i\};x,x^i)$ and $R_\omega(\hat{u}\{\tilde{\omega}^i\};x,x^i)$ in terms of the set of unknowns $\hat{u}\{\tilde{\omega}^i\}:=\{\hat{u}(x^j),\ x^j\in\tilde{\omega}^i\}$ in (51), (52) and (54) are obtained after substitution of interpolation formulae (61) for u in the coefficient $a(u;\cdot)$ participating in the definitions for $T(u)$, $P_\omega(u;x,y)$ and $R_\omega(u;x,y)$ in Section 2. The right-hand side components $\mathcal{F}_\omega(\hat{u}\{\tilde{\omega}^i\},x^i)$ and $\mathcal{F}^0_\omega(\hat{u}\{\tilde{\omega}^i\},x^i)$ are obtained similarly after using interpolation formulae (61), for u in (21) and (23).

4.2.3. *Mesh-less discretization of quasi-linear two-operator direct BDIDEs*
Partly segregated formulation. After substituting approximations (61) and (62) in LBDIDE (44) and boundary condition (27), we arrive at the following system of $J+J_D$ quasi-linear algebraic equations with respect to the J unknowns $\hat{u}(x^j)$, $x_j\in\bar{\Omega}$ and J_D unknowns $\hat{t}(x^j)$, $x_j\in\partial_D\Omega$,

$$\sum_{x^j\in\tilde{\omega}^i}\left[c^0(x^i)\Phi_j(x^i)+K^0_{ij}(\hat{u}\{\tilde{\omega}^i\})\right]\hat{u}(x^j)+\sum_{x^j\in\partial_D\Omega\cap\tilde{\omega}^i}Q_{ij}(\hat{u}\{\tilde{\omega}^i\})\hat{t}\left(x^j\right)$$
$$=\mathcal{F}^0_\omega(\hat{u}\{\tilde{\omega}^i\},x^i),\quad x^i\in\Omega\cup\partial\Omega, \tag{66}$$
$$\sum_{x^j\in\tilde{\omega}^i}\hat{u}(x^j)\Phi_j(x^i)=\bar{u}(x^i),\quad x^i\in\partial_D\Omega,\qquad\text{no sum in } i. \tag{67}$$

United formulation. Alternatively, one can arrive at another quasi-linear system of J algebraic equations with respect to J unknowns $\hat{u}(x^j)$, $x^j\in\bar{\Omega}$, by substituting approximation formulae (61) in LBDIDE (42),

$$\sum_{x^j\in\tilde{\omega}^i}\left[c(x^i)\Phi_j(x^i)+K_{ij}(\hat{u}\{\tilde{\omega}^i\})\right]\hat{u}(x^j)=\mathcal{F}_\omega(\hat{u}\{\tilde{\omega}^i\},x^i),\quad x^i\in\bar{\Omega},\quad\text{no sum in } i. \tag{68}$$

The matrices K^0_{ij}, Q_{ij}, K_{ij} in (66) and (68) are expressed by (56), (57) and (59) with the shape functions Φ_j, Φ'_j from (61), (62), and $u\{\tilde{\omega}^i\}$ replaced by $\hat{u}\{\tilde{\omega}^i\}$. The expressions for $T(\hat{u}\{\tilde{\omega}^i\})$, $T^{(x^i)}(\hat{u}\{\tilde{\omega}^i\})$, $P^{(x^i)}_\omega(\hat{u}\{\tilde{\omega}^i\};x,x^i)$ and $R^{(x^i)}_\omega(\hat{u}\{\tilde{\omega}^i\};x,x^i)$ in terms of the set of unknowns $\hat{u}\{\tilde{\omega}^i\}:=\{u(x^j),\ x^j\in\tilde{\omega}^i\}$ in (56), (57) and (59) are obtained after substituting interpolation formulae (61) for u in the coefficient $a(u;\cdot)$ in the definitions for $T(u)$, $T^{(y)}(u)$, $P^{(y)}_\omega(u;x,y)$ and $R^{(y)}_\omega(u;x,y)$ in Section 3. The right-hand side components $\mathcal{F}_\omega(\hat{u}\{\tilde{\omega}^i\},x^i)$ and $\mathcal{F}^0_\omega(\hat{u}\{\tilde{\omega}^i\},x^i)$ in (66) and (68) are obtained similarly after using interpolation formulae (61), for u in (43) and (45).

Sparsity. From the definitions in both mesh-based and mesh-less methods, we have $\Phi_j(x)=\nabla\Phi_j(x)=[T(u)\Phi_j](x)=[T^{(y)}(u)\Phi_j](x)=\Phi'_j(x)=0$ if $x\in\bar{\omega}(x^i)$ but $x^j\notin\tilde{\omega}^i$ and consequently $K^0_{ij}=Q_{ij}=K_{ij}=0$ if $x^j\notin\tilde{\omega}^i$. In addition, K^0_{ij}, Q_{ij}, K_{ij} depend only on $u\{\tilde{\omega}^i\}$ or $\hat{u}\{\tilde{\omega}^i\}$, respectively. Thus, each equation in (50), (53), (55), (58) and (63–68) has not more than $J_{\tilde{\omega}^i}\ll J$ non-zero entries, *i.e.*, *the systems are sparse.*

5. Concluding remarks

The parametrix localization by multiplication by a cut-off function with local support allows us to reduce a BVP for a second-order quasi-linear PDE to a direct or two-operator direct localized quasi-linear boundary-domain integro-differential equation. The equation includes at most the first derivative of the unknown solution, weakly singular integrals over the domain, and at most Cauchy-type singular integrals over the boundary.

Examples of different cut-off functions with different smoothness leading to different LBDIDEs demonstrate the flexibility of the method. Algorithms of both mesh-based and mesh-less discretization of LBDIDEs leading to sparse systems of quasi-linear algebraic equations, similar to FEM, show the great potential of the LBDIDE method for the numerical solution of different BVPs in science and engineering.

For each mixed BVP, united and partly segregated formulations are presented (coinciding for the pure Neumann problem). The first leads to BDIDEs of the second kind, which look promising for constructing simple and fast-converging iteration algorithms of their solution.

Investigation of the equivalence of the BDIDEs to the original BVPs, solvability, uniqueness of solution, and convergence of the iteration algorithm, including analysis of spectral properties of the corresponding linear BDIDEs, needs to be done for constructing robust numerical methods based on this information [16] and for an optimal choice of the cut-off functions, localization domains and node points.

References

1. C.A. Brebbia, J.C.F. Telles and L.C. Wrobel, *Boundary Element Techniques.* Berlin: Springer (1984) 464pp.
2. J. Balaš, J. Sladek and V. Sladek, *Stress Analysis by Boundary Element Methods*. Amsterdam – Oxford – New York – Tokyo: Elsevier (1989) 686 pp.
3. P.K. Banerjee, *The Boundary Element Methods in Engineering*. London: McGraw-Hill (1994) 496 pp.
4. L.C. Wrobel, *The Boundary Element Method, Vol. 1, Applications in Thermo-Fluids and Acoustics*. Chichester: Wiley (2002) 451 pp.
5. M.H. Aliabadi, *The Boundary Element Method, Vol. 2, Applications in Solids and Structures*. Chichester: Wiley (2002) 580 pp.
6. S.E. Mikhailov, Localized boundary-domain integral formulations for problems with variable coefficients. *Engng. Anal. Bound. Elem.* 26 (2002) 681–690.
7. S.E. Mikhailov and I.S. Nakhova, Numerical solution of a Neumann problem with variable coefficients by the localized boundary-domain integral equation method. In: S. Amini (ed.), *Fourth UK Conference on Boundary Integral Methods*, ISBN 0-902896-40-7, Salford: Salford University (2003) pp. 175–184.
8. S.E. Mikhailov, About localized boundary-domain integro-differential formulations for a quasi-linear problem with variable coefficients. In: C. Constanda, M. Ahues and A. Largillier (eds), *Integral Methods in Science and Engineering: Analytic and Numerical Techniques*. Boston – Basel – Berlin: Birkhäuser, ISBN 0-8176-3228-X (2004) pp. 139–144.
9. S.E. Mikhailov, Some localized boundary-domain integro-differential formulations for quasi-linear problems with variable coefficients. In: R.Gallego and M.H.Aliabadi (eds), *Advances in Boundary Element Techniques IV*. ISBN 0904 188965, London: Queen Mary, University of London (2003) pp. 289–294.
10. C. Miranda, *Partial Differential Equations of Elliptic Type.* Berlin–Heidelberg–New York: Springer (1970) 370 pp.
11. T. Zhu, J.-D. Zhang and S.N. Atluri, A local boundary integral equation (LBIE) method in computational mechanics, and a meshless discretization approach. *Comp. Mech.* 21 (1998) 223–235.
12. T. Zhu, J.-D. Zhang and S.N. Atluri, A meshless numerical method based on the local boundary integral equation (LBIE) to solve linear and non-linear boundary value problems. *Engng. Anal. Bound. Elem.* 23 (1999) 375–389.
13. A. Pomp, Levi functions for linear elliptic systems with variable coefficients including shells. *Comp. Mech.* 22 (1998) 93–99.
14. A. Pomp, *The Boundary-Domain Integral Methods for Elliptic Systems (With an Application to Shells)*. Berlin, Heidelberg: Springer (1998) 164 pp.
15. T. Belytschko, Y. Krongauz, D. Organ, M. Flemming and P. Krysl, Meshless methods: An overview and recent developments. *Comp. Methods Appl. Mech. Engng.* 139 (1996) 4–47.
16. S.E. Mikhailov, Finite-dimensional perturbations of linear operators and some applications to boundary integral equations. *Engng. Anal. Bound. Elem.* 23 (1999) 805–813.